# A Cyclic Polarized Isogeny on a Simple Abelian Threefold

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**Abstract**

We construct a simple complex abelian threefold $A$ and a polarized endomorphism $f : A \to A$ such that $f$ is not incompressible, in the sense of essential dimension. More precisely, $f$ is 4-polarized, $\deg f = 64$, and $\ker(f) \simeq \mathbb{Z}/64\mathbb{Z}$. Since a cyclic cover over $\mathbb{C}$ has essential dimension at most one, while $\dim A = 3$, this gives a concrete negative answer to the question whether every polarized endomorphism of a simple abelian variety is incompressible.

## 1 Definitions and main result

Throughout the paper the base field is $\mathbb{C}$. If $\pi : Y \dashrightarrow X$ is a generically finite dominant rational map of irreducible varieties, its essential dimension is denoted by $\mathrm{ed}(\pi)$. We call $\pi$ *incompressible* if

$$\mathrm{ed}(\pi) = \dim X = \dim Y.$$

Equivalently, $\pi$ cannot be birationally obtained by base change from a generically finite map of smaller-dimensional varieties.

We shall use the following standard fact. If $\pi$ is a finite abelian Galois cover with Galois group $G$, then

$$\mathrm{ed}(\pi) \leq \mathrm{rank}(G),$$

where $\mathrm{rank}(G)$ is the minimum number of generators of $G$. In particular, a cyclic cover over $\mathbb{C}$ has essential dimension at most one. Indeed, by Kummer theory, a cyclic degree-$n$ extension of a field containing the $n$-th roots of unity is obtained from the one-dimensional cover

$$\mathbb{G}_m \longrightarrow \mathbb{G}_m, \qquad t \longmapsto t^n.$$

**Theorem 1.** *There exists a simple abelian threefold $A$ over $\mathbb{C}$ and a polarized endomorphism $f : A \to A$ such that $f$ is not incompressible. More precisely, there is a 4-polarized isogeny $f$ with*

$$\ker(f) \simeq \mathbb{Z}/64\mathbb{Z}, \qquad \mathrm{ed}(f) = 1 < 3 = \dim A.$$

## 2 The CM abelian threefold

Let

$$\zeta = \zeta_7 = e^{2\pi i/7}, \qquad K = \mathbb{Q}(\zeta).$$

Then $K$ is a CM field of degree 6. Let

$$\sigma_j : K \hookrightarrow \mathbb{C}, \qquad \sigma_j(\zeta) = \zeta^j, \qquad j = 1, 2, 3,$$

and choose the CM type

$$\Phi = \{\sigma_1, \sigma_2, \sigma_3\}.$$

Put

$$\alpha = 2\zeta, \qquad R = \mathbb{Z}[\alpha] = \mathbb{Z}[2\zeta].$$

Via the CM type $\Phi$, regard $R$ as a lattice in $\mathbb{C}^3$ by

$$x \longmapsto \Phi(x) = (\sigma_1(x), \sigma_2(x), \sigma_3(x)).$$

Define the complex torus

$$A = \mathbb{C}^3/\Phi(R).$$

## 2.1 Projectivity

We first show that $A$ is an abelian variety. Let

$$\delta = \zeta^{-1} - \zeta.$$

Then $\overline{\delta} = -\delta$. For $x, y \in R$, define

$$E(x, y) = \mathrm{Tr}_{K/\mathbb{Q}}(\delta x \overline{y}).$$

Since $\delta, x, \overline{y}$ are algebraic integers, $E(x, y) \in \mathbb{Z}$. Moreover $E$ is alternating, because

$$E(y, x) = \mathrm{Tr}_{K/\mathbb{Q}}(\delta y \overline{x}) = \mathrm{Tr}_{K/\mathbb{Q}}(\overline{\delta y \overline{x}}) = \mathrm{Tr}_{K/\mathbb{Q}}(-\delta x \overline{y}) = -E(x, y).$$

For $j = 1, 2, 3$ we have

$$\operatorname{Im} \sigma_j(\delta) = -2\sin(2\pi j/7) < 0.$$

Thus $E$ is the imaginary part of the positive definite Hermitian form

$$H(z, w) = \sum_{j=1}^{3} c_j z_j \overline{w_j}, \qquad c_j = -2 \operatorname{Im} \sigma_j(\delta) > 0.$$

Indeed, for $x, y \in K$ one checks that

$$E(x, y) = \sum_{j=1}^{3} c_j \operatorname{Im}\big(\sigma_j(x)\overline{\sigma_j(y)}\big).$$

Hence $E$ is a Riemann form on the torus $A$, and $A$ is projective. Therefore $A$ is an abelian variety.

We shall use a slightly scaled line bundle. Since $E$ is integral, the form $E_0 = 2E$ is even integral. By the Appell–Humbert theorem, the Riemann form $E_0$ together with the trivial semicharacter defines an ample line bundle $L$ on $A$.

### 2.2 Simplicity

**Lemma 1.** *The abelian variety $A$ is simple.*

*Proof.* The lattice $R = \mathbb{Z}[2\zeta]$ has finite index in the full ring of integers $\mathcal{O}_K = \mathbb{Z}[\zeta]$. Hence $A$ is isogenous to

$$A_0 = \mathbb{C}^3/\Phi(\mathcal{O}_K).$$

It is therefore enough to prove that $A_0$ is simple.

The Galois group of $K/\mathbb{Q}$ is

$$\mathrm{Gal}(K/\mathbb{Q}) \simeq (\mathbb{Z}/7\mathbb{Z})^\times.$$

The only proper CM subfield of $K$ is the quadratic subfield $\mathbb{Q}(\sqrt{-7})$. The subgroup fixing it is

$$H = \{1, 2, 4\} \subset (\mathbb{Z}/7\mathbb{Z})^\times.$$

Therefore the CM types of $K$ induced from this quadratic CM subfield are precisely

$$\{1, 2, 4\} \qquad \text{and} \qquad \{3, 5, 6\}.$$

Our CM type is

$$\Phi = \{1, 2, 3\},$$

which is neither of these. Thus $\Phi$ is primitive. By the standard criterion for complex multiplication, a CM abelian variety with primitive CM type is simple. Hence $A_0$ is simple, and since simplicity is invariant under isogeny, $A$ is simple. □

## 3 The polarized endomorphism

Multiplication by

$$\alpha = 2\zeta$$

preserves the lattice $R = \mathbb{Z}[\alpha]$. It therefore defines an endomorphism

$$f : A \longrightarrow A, \qquad z \longmapsto \alpha z,$$

where on the universal cover $\mathbb{C}^3$ this means diagonal multiplication by

$$(\sigma_1(\alpha), \sigma_2(\alpha), \sigma_3(\alpha)).$$

Since $\alpha \neq 0$, this endomorphism is an isogeny.

**Lemma 2.** *The endomorphism $f$ is 4-polarized with respect to the ample line bundle $L$ constructed above. In particular,*

$$\deg(f) = 4^3 = 64.$$

*Proof.* For $x, y \in R$,

$$\begin{aligned} E(\alpha x, \alpha y) &= \mathrm{Tr}_{K/\mathbb{Q}}(\delta \alpha x \overline{\alpha y}) \\ &= \mathrm{Tr}_{K/\mathbb{Q}}(\delta \alpha \overline{\alpha} x \overline{y}). \end{aligned}$$

But

$$\alpha\overline{\alpha} = (2\zeta)(2\zeta^{-1}) = 4.$$

Therefore

$$E(\alpha x, \alpha y) = 4E(x, y), \qquad E_0(\alpha x, \alpha y) = 4E_0(x, y).$$

The line bundle $L$ was defined by the Riemann form $E_0$ and the trivial semicharacter. Pulling back by $f$ multiplies the Riemann form by 4, and the trivial semicharacter remains trivial. Thus, by the Appell–Humbert theorem,

$$f^*L \simeq L^{\otimes 4}.$$

So $f$ is 4-polarized. Since $A$ has dimension 3, a 4-polarized endomorphism has degree $4^3 = 64$. □

**Lemma 3.** *The kernel of $f$ is cyclic of order* 64*:*

$$\ker(f) \simeq \mathbb{Z}/64\mathbb{Z}.$$

*Proof.* The kernel is

$$\ker(f) \simeq \alpha^{-1}R/R.$$

Multiplication by $\alpha$ identifies this group with $R/\alpha R$.

The minimal polynomial of $\alpha = 2\zeta$ is

$$\begin{aligned} P(t) &= 2^6\Phi_7(t/2) \\ &= t^6 + 2t^5 + 4t^4 + 8t^3 + 16t^2 + 32t + 64. \end{aligned}$$

Hence

$$R = \mathbb{Z}[\alpha] \simeq \mathbb{Z}[t]/(P(t)).$$

Modulo the ideal generated by $\alpha$, equivalently by $t$, we get

$$\begin{aligned} R/\alpha R &\simeq \mathbb{Z}[t]/(P(t), t) \\ &\simeq \mathbb{Z}/(P(0)) \\ &\simeq \mathbb{Z}/64\mathbb{Z}. \end{aligned}$$

Thus $\ker(f)$ is cyclic of order 64. □

# 4 Failure of incompressibility

The isogeny $f : A \to A$ is finite etale and Galois; its Galois group is the translation group by its kernel:

$$\mathrm{Gal}(f) \simeq \ker(f) \simeq \mathbb{Z}/64\mathbb{Z}.$$

Since this group is cyclic, the essential dimension of the cover is at most one:

$$\mathrm{ed}(f) \leq 1.$$

On the other hand $f$ has degree $64 > 1$, so it is not birational. A nontrivial finite cover over an algebraically closed base field cannot have essential dimension zero. Hence

$$\mathrm{ed}(f) = 1.$$

But

$$\dim A = 3.$$

Therefore

$$\operatorname{ed}(f) = 1 < 3 = \dim A,$$

and $f$ is not incompressible. This proves Theorem 1.

## 5 Where the hypotheses are used

(1) The fact that $A$ is an abelian variety is obtained from the explicit Riemann form

$$E(x, y) = \operatorname{Tr}_{K/\mathbb{Q}}((\zeta^{-1} - \zeta)x\overline{y}).$$

This is the projectivity input.

(2) The simplicity of $A$ comes from the primitive CM type

$$\Phi = \{1, 2, 3\} \subset (\mathbb{Z}/7\mathbb{Z})^{\times}.$$

The only proper CM subfield of $\mathbb{Q}(\zeta_7)$ is $\mathbb{Q}(\sqrt{-7})$, and the types induced from it are $\{1, 2, 4\}$ and $\{3, 5, 6\}$, not $\{1, 2, 3\}$.

(3) The polarizedness of $f$ uses the identity

$$(2\zeta)(2\zeta^{-1}) = 4.$$

It implies that multiplication by $2\zeta$ multiplies the chosen Riemann form by 4, hence $f^*L \simeq L^{\otimes 4}$.

(4) The failure of incompressibility uses the non-maximal order

$$R = \mathbb{Z}[2\zeta]$$

rather than the full ring of integers. This choice makes

$$R/(2\zeta)R \simeq \mathbb{Z}/64\mathbb{Z},$$

so the Galois group of the isogeny is cyclic. A cyclic cover has essential dimension at most one, which is strictly smaller than $\dim A = 3$.

*Remark* 1. The example is constructed analytically over $\mathbb{C}$. Since it is a CM abelian variety with CM endomorphism $2\zeta$, the usual theory of complex multiplication also gives models over $\overline{\mathbb{Q}}$ after replacing it by an isomorphic algebraic model.

## References


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