PICARD NUMBER ONE FANO MANIFOLDS ADMITTING GALOIS SELF-COVERS

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Abstract. Let X be a smooth Fano manifold over $\mathbb{C}$ with Picard number one, and let $f\colon X \to X$ be a polarized Galois endomorphism of degree greater than one. Write $f^*H \sim qH$ for the primitive ample generator H of $\mathrm{Pic}(X)$, and let $G = \mathrm{Gal}(f)$. If $B_1, \dots, B_r$ are the divisorial components of the branch locus and e_j is the generic ramification index over B_j, set $E(f) = \prod_j e_j$. We prove the index estimate

$$i_X \geq \log_q E(f) \geq \log_q |G^{\mathrm{ab}}|,$$

where i_X is the Fano index and $G^{\mathrm{ab}} = G/[G,G]$. Consequently, if $E(f) > q^{n-2}$, then $X \simeq \mathbb{P}^n$; in particular it is enough to assume $|G^{\mathrm{ab}}| > q^{n-2}$, equivalently $|[G,G]| < q^2$. As a four-dimensional application of the same Galois branch formalism, we also prove that every smooth Fano fourfold of Picard number one admitting a polarized Galois endomorphism of degree greater than one is isomorphic to $\mathbb{P}^4$.

1. Introduction

A guiding expectation in the study of endomorphisms of Fano manifolds is that a smooth Fano manifold of Picard number one should admit a non-isomorphic surjective endomorphism only in the projective-space case. The purpose of this paper is to record a group-theoretic strengthening of the abelian Galois case and to combine it with the low-dimensional ramification argument in dimension four.

Let X be a smooth Fano manifold over $\mathbb{C}$ of dimension n and Picard number $\rho(X) = 1$. An endomorphism $f\colon X \to X$ is called *polarized* if there is an ample divisor H and an integer $q > 1$ such that

$$f^*H \sim qH.$$

It is called *Galois* if the finite extension $\mathbb{C}(X)/f^*\mathbb{C}(X)$ is Galois. After replacing H by the primitive ample generator of $\mathrm{Pic}(X)$, the integer q is uniquely determined; moreover $\deg f = q^n$. Let $G = \mathrm{Gal}(f)$. For the prime divisorial components $B_1, \dots, B_r$ of the branch locus of f, let e_j denote the ramification index over the generic point of B_j, and put

$$E(f) := \prod_{j=1}^{r} e_j.$$

The first main result is the following large-ramification criterion.

Theorem A (Large ramification product or large abelian quotient)**.** *Let X be a smooth Fano n-fold over $\mathbb{C}$ with $\rho(X) = 1$, and let $f\colon X \to X$ be a polarized Galois endomorphism of degree greater than one. With the notation above, if*

$$E(f) > q^{n-2},$$

then

$$X \simeq \mathbb{P}^n.$$

In particular, the same conclusion holds if

$$|G^{\mathrm{ab}}| > q^{n-2}.$$

Since $|G| = q^n$, this last condition is equivalent to

$$|[G,G]| < q^2.$$

The numerical input is the index estimate

$$i_X \geq \log_q E(f) \geq \log_q |G^{\mathrm{ab}}|.$$

Thus the proof does not require G to be abelian. It only requires that enough codimension-one ramification survive in the abelianization of G. When G is abelian, $G^{\mathrm{ab}} = G$ and $|G| = q^n$, so Theorem A recovers the usual abelian Galois conclusion.

Our second principal result is the four-dimensional case of the low-dimensional rigidity theorem for polarized Galois endomorphisms.

Theorem B (The four-dimensional application)**.** *Let X be a smooth Fano fourfold over $\mathbb{C}$ with $\rho(X) = 1$. If X admits a polarized Galois endomorphism $f\colon X \to X$ of degree greater than one, then*

$$X \simeq \mathbb{P}^4.$$

Theorem A proves Theorem B whenever $E(f) > q^2$, or equivalently under the sufficient large-abelian-quotient condition $|G^{\mathrm{ab}}| > q^2$. The remaining four-dimensional cases are excluded by a short branch-counting argument in Fano index one. The proof uses two standard rigidity inputs: the Kobayashi–Ochiai high-index theorem, together with the dynamical rigidity results excluding non-isomorphic polarized endomorphisms of smooth quadrics, del Pezzo manifolds, and Mukai manifolds of Picard number one.

2. Basic facts on Picard groups and Galois covers

We first collect the standard facts needed later. All varieties are defined over $\mathbb{C}$.

Lemma 2.1 (The Picard group)**.** *Let X be a smooth Fano manifold with $\rho(X) = 1$. Then*

$$\mathrm{Pic}(X) \simeq \mathbb{Z}[H]$$

for a primitive ample divisor H.

Proof. Kodaira vanishing gives $H^1(X, \mathcal{O}_X) = 0$, hence $\mathrm{Pic}^0(X) = 0$ and $\mathrm{Pic}(X) = \mathrm{NS}(X)$ is a finitely generated abelian group of rank one. It remains to exclude torsion. Smooth Fano manifolds over $\mathbb{C}$ are rationally connected by Campana and by Kollár–Miyaoka–Mori, and smooth rationally connected complex projective varieties are simply connected; see [1, 2, 3]. If $L \in \mathrm{Pic}(X)$ were a non-trivial torsion line bundle of exact order $m > 1$, then

$$\mathrm{Spec}_X(\mathcal{O}_X \oplus L^{-1} \oplus \cdots \oplus L^{-(m-1)}) \longrightarrow X$$

would be a connected cyclic finite étale cover of degree m, contradicting simple connectedness. Therefore $\mathrm{Pic}(X)$ is torsion-free of rank one, and its generator can be chosen ample. $\square$

Lemma 2.2 (Polarization)**.** *Let X be as in Lemma 2.1, and let $f\colon X \to X$ be a polarized endomorphism. Then*

$$f^*H \sim qH$$

for an integer $q > 1$, the morphism f is finite, and

$$\deg f = q^n.$$

Proof. The existence of an integer $q > 1$ follows from the definition of polarization and from Lemma 2.1. Since f^*H is ample, no curve can be contracted by f: if a curve C were contracted, then $(f^*H \cdot C) = 0$, contradicting the ampleness of f^*H. Thus f is finite. Finally,

$$(f^*H)^n = (\deg f)H^n,$$

and the left-hand side is $q^n H^n$, so $\deg f = q^n$. □

Lemma 2.3 (Galois quotient and branch formula)**.** *Let X be a smooth Fano manifold with $\rho(X) = 1$, and let $f\colon X \to X$ be a polarized Galois endomorphism of degree greater than one. Let $G = \mathrm{Gal}(f)$. Then, after identifying the target copy of X with the quotient X/G, the morphism f is the quotient morphism $X \to X/G$, and $|G| = q^n$.*

Let $B_1, \dots, B_r$ be the irreducible components of the branch divisor on the target, and let

$$F_j = (f^{-1}B_j)_{\mathrm{red}}.$$

Then $r \geq 1$ and there is an integer $e_j > 1$ such that

$$f^*B_j = e_j F_j.$$

Moreover, if

$$\Delta_f := \sum_{j=1}^{r} \left(1 - \frac{1}{e_j}\right) B_j,$$

then

$$K_X = f^*(K_X + \Delta_f), \qquad \Delta_f \sim_{\mathbb{Q}} \frac{q-1}{q}(-K_X).$$

Proof. The Galois hypothesis identifies $\mathbb{C}(X)/f^*\mathbb{C}(X)$ with a finite Galois extension. Since X is normal, the deck transformations are regular, and f is the quotient by a finite group G of order $\deg f = q^n$.

The morphism is finite and generically étale. If it were étale, then it would be a connected finite étale cover of the simply connected Fano manifold X, hence of degree one, contrary to $\deg f > 1$. Thus the branch locus is non-empty; by purity of the branch locus it is divisorial [8, Tag 0BMB]. At the generic point of a divisor over a branch component, the completed local form is $s = t^e$ in characteristic zero. The Galois group acts transitively over the generic point of each B_j, so the ramification index is independent of the prime divisor lying over B_j. Hence $f^*B_j = e_j F_j$, and the ramification divisor is

$$R_f = \sum_{j=1}^{r} (e_j - 1)F_j = f^*\Delta_f.$$

The Hurwitz formula $K_X = f^*K_X + R_f$ gives $K_X = f^*(K_X + \Delta_f)$. Writing $-K_X \sim i_X H$ and $\Delta_f \sim_{\mathbb{Q}} aH$, and using $f^*H \sim qH$, we obtain

$$-i_X = q(-i_X + a),$$

so $a = i_X(q-1)/q$. This is exactly $\Delta_f \sim_{\mathbb{Q}} \frac{q-1}{q}(-K_X)$. □

Lemma 2.4 (Inertia generation and abelianization)**.** *Let $\pi\colon Y \to Z$ be a finite Galois morphism of normal complex projective varieties with Galois group G. Assume that Z is smooth and simply connected. Let $D_1, \dots, D_r$ be the prime divisorial components of the branch locus of π. For each j, choose a prime divisor $E_j \subset Y$ over D_j, and let $I_j \subset G$ be the inertia subgroup* *at the generic point of E_j. Put $e_j = |I_j|$. Then the divisorial inertia subgroups generate G as a normal subgroup. In particular, the images of $I_1, \dots, I_r$ generate G^{ab}, and*

$$|G^{\mathrm{ab}}| \leq \prod_{j=1}^{r} e_j.$$

Proof. Since the ground field has characteristic zero, the extensions of discrete valuation rings at the generic points of the divisors are tamely ramified, and the order of the inertia subgroup is the corresponding ramification index. Let $N \triangleleft G$ be the normal subgroup generated by the subgroups $I_1, \dots, I_r$. Inertia groups over the same branch divisor are conjugate; hence N contains every codimension-one inertia subgroup of Y over Z.

Consider the intermediate quotient

$$Y \longrightarrow Y/N \longrightarrow Y/G \simeq Z.$$

The finite morphism $Y/N \to Z$ has trivial inertia at every codimension-one point: outside the branch locus there is no inertia, and over each branch divisor the inertia has been killed by quotienting by N. Therefore $Y/N \to Z$ is unramified in codimension one. By Zariski–Nagata purity of the branch locus it is finite étale [8, Tag 0BMB]. Since Z is simply connected and Y/N is connected, this cover has degree one. Thus G/N is trivial, so $N = G$.

Passing to the abelianization kills conjugation. Hence the images of $I_1, \dots, I_r$ generate G^{ab}, and the product of these images maps surjectively onto G^{ab}. Since the image of I_j has order at most $|I_j| = e_j$, we obtain $|G^{\mathrm{ab}}| \leq \prod_j e_j$. □

3. The index estimate

The next elementary inequality is the numerical point in the proof.

Lemma 3.1. *Let $q > 1$ and $e \geq 2$ be integers, and put $g = \gcd(q, e)$. Then*

$$\frac{q}{g}(e-1) \geq (q-1)\log_q(e).$$

Proof. First suppose $e \leq q$. Since $g \leq e$, it is enough to prove

$$\frac{q(e-1)}{e} \geq (q-1)\log_q(e).$$

Equivalently,

$$\frac{q\log q}{q-1} \geq \frac{e\log e}{e-1}.$$

The function

$$\Phi(t) = \frac{t\log t}{t-1} \qquad (t > 1)$$

is increasing, because

$$\Phi'(t) = \frac{t-1-\log t}{(t-1)^2} > 0.$$

Thus the desired inequality follows from $e \leq q$.

Now suppose $e \geq q$. Since $g \leq q$, the left-hand side is at least $e-1$. Let

$$\Psi(t) = t - 1 - (q-1)\log_q(t) \qquad (t \geq q).$$

Then $\Psi(q) = 0$, and

$$\Psi'(t) = 1 - \frac{q-1}{t\log q} > 0$$

for $t \geq q$, because $q\log q > q-1$. Hence $e-1 \geq (q-1)\log_q(e)$, which proves the lemma. □

Theorem 3.2 (Index estimate)**.** *Let X be a smooth Fano n-fold with $\rho(X) = 1$, and let $f\colon X \to X$ be a polarized Galois endomorphism of degree greater than one. With the notation above,*

$$i_X \geq \log_q E(f) \geq \log_q |G^{\mathrm{ab}}|,$$

where i_X is the Fano index of X.

Proof. By Lemma 2.1, write $\operatorname{Pic}(X) = \mathbb{Z}[H]$ with H primitive and ample. By Lemma 2.2, $f^*H \sim qH$ and $\deg f = q^n$. Write

$$-K_X \sim i_X H.$$

Let $B_1, \dots, B_r$ be the prime divisorial components of the branch locus of f, and set $F_j = (f^{-1}B_j)_{\mathrm{red}}$. By Lemma 2.3,

$$f^*B_j = e_j F_j.$$

Since $\operatorname{Pic}(X) = \mathbb{Z}[H]$, there are positive integers a_j, b_j such that

$$F_j \sim a_j H, \qquad B_j \sim b_j H.$$

Pulling back $B_j \sim b_j H$ gives

$$q b_j = e_j a_j.$$

Let $g_j = \gcd(q, e_j)$. The last equality implies that a_j is divisible by q/g_j, so

$$a_j \geq \frac{q}{g_j}. \tag{3.1}$$

The ramification divisor is

$$R_f = \sum_{j=1}^{r} (e_j - 1) F_j.$$

The Hurwitz formula $K_X = f^*K_X + R_f$ gives

$$i_X(q-1) = \sum_{j=1}^{r} a_j(e_j - 1). \tag{3.2}$$

Using (3.1) and Lemma 3.1, we obtain

$$\begin{aligned} i_X(q-1) &= \sum_{j=1}^{r} a_j(e_j - 1) \\ &\geq \sum_{j=1}^{r} \frac{q}{g_j}(e_j - 1) \\ &\geq \sum_{j=1}^{r} (q-1)\log_q(e_j) \\ &= (q-1)\log_q\left(\prod_{j=1}^{r} e_j\right). \end{aligned}$$

Since $q > 1$, this yields $i_X \geq \log_q E(f)$. Finally, Lemma 2.4, applied to $f\colon X \to X$, gives $|G^{\mathrm{ab}}| \leq E(f)$, hence

$$i_X \geq \log_q E(f) \geq \log_q |G^{\mathrm{ab}}|.$$

□

4. Proof of the large-ramification theorem

We now prove Theorem A.

Proof of Theorem A. If $n = 1$, then $X \simeq \mathbb{P}^1$. If $n = 2$, then X is a smooth del Pezzo surface with Picard number one; by the classical classification of smooth del Pezzo surfaces over $\mathbb{C}$, $X \simeq \mathbb{P}^2$. Assume from now on that $n \geq 3$.

If $E(f) > q^{n-2}$, then Theorem 3.2 gives

$$i_X > n - 2.$$

Since i_X is an integer, $i_X \geq n-1$. By the theorem of Kobayashi–Ochiai [4], the cases $i_X = n+1$ and $i_X = n$ give respectively $X \simeq \mathbb{P}^n$ and $X \simeq Q^n \subset \mathbb{P}^{n+1}$, where Q^n is a smooth quadric. If $X \simeq Q^n$ and $n \geq 3$, then every finite surjective endomorphism of Q^n is an automorphism by Paranjape–Srinivas [5, Proposition 8], contradicting $\deg f = q^n > 1$.

It remains to consider $i_X = n-1$. Then X is a del Pezzo manifold of Picard number one. The dynamical rigidity results of Shao–Zhong exclude non-isomorphic surjective endomorphisms in this case [6]. Thus the only possible case is $X \simeq \mathbb{P}^n$.

The assertion involving the abelianization follows because Theorem 3.2 gives $|G^{\mathrm{ab}}| \leq E(f)$. Finally, since $|G| = \deg f = q^n$, we have

$$|G^{\mathrm{ab}}| = \frac{|G|}{|[G,G]|} = \frac{q^n}{|[G,G]|}.$$

Therefore $|G^{\mathrm{ab}}| > q^{n-2}$ is equivalent to $|[G,G]| < q^2$. ☐

Corollary 4.1 (The abelian Galois case)**.** *Let X be a smooth Fano n-fold over $\mathbb{C}$ with $\rho(X) = 1$, and let $f \colon X \to X$ be a polarized Galois endomorphism of degree greater than one. If $G = \mathrm{Gal}(f)$ is abelian, then $X \simeq \mathbb{P}^n$.*

Proof. If G is abelian, then $G^{\mathrm{ab}} = G$. Hence

$$|G^{\mathrm{ab}}| = |G| = \deg f = q^n > q^{n-2},$$

and Theorem A applies. ☐

5. The four-dimensional application

Theorem A already proves the fourfold result under the numerical condition $E(f) > q^2$. In this section we remove that extra hypothesis in dimension four. The proof uses the same branch formula and inertia-generation mechanism, together with standard high-index rigidity.

Proposition 5.1 (High-index reduction)**.** *Let X be a smooth Fano n-fold over $\mathbb{C}$ with $\rho(X) = 1$, and let $f \colon X \to X$ be a polarized Galois endomorphism of degree greater than one. If $n \geq 3$ and $i_X \geq n-2$, then $X \simeq \mathbb{P}^n$.*

Proof. If $i_X \geq n-1$, the same argument used in the proof of Theorem A applies: Kobayashi–Ochiai leaves only projective space and smooth quadrics at indices $n+1$ and n, while the case $i_X = n-1$ is a del Pezzo manifold of Picard number one. Smooth quadrics of dimension at least three admit no non-isomorphic finite self-maps by Paranjape–Srinivas [5, Proposition 8], and del Pezzo manifolds of Picard number one are dynamically rigid by Shao–Zhong [6]. Hence only $\mathbb{P}^n$ can occur.

It remains to treat $i_X = n-2$. Then X is a Mukai manifold of Picard number one. Shao–Zhong's dynamical rigidity theorem for Mukai manifolds of Picard number one excludes non-isomorphic surjective endomorphisms in this case [7]. This contradicts $\deg f > 1$ unless $X \simeq \mathbb{P}^n$. ☐

Lemma 5.2 (Exclusion of Fano index one)**.** *Let X be a smooth Fano n-fold over $\mathbb{C}$ with $\rho(X) = 1$, where $n \geq 2$. Let $f\colon X \to X$ be a polarized Galois endomorphism of degree greater than one. Then $i_X \neq 1$.*

Proof. Assume, for contradiction, that $i_X = 1$. Then $-K_X \sim H$. Use the notation of Lemma 2.3. Write

$$B_j \sim b_j H$$

with b_j a positive integer. Since $\Delta_f \sim_{\mathbb{Q}} \frac{q-1}{q}(-K_X)$, we have

$$\sum_{j=1}^{r} \left(1 - \frac{1}{e_j}\right) b_j = \frac{q-1}{q}. \tag{5.1}$$

For every j, one has $e_j > 1$, hence $1 - 1/e_j \geq 1/2$, and $b_j \geq 1$. The right-hand side of (5.1) is strictly smaller than 1. Therefore there is exactly one branch component, say B_1, and $b_1 = 1$. Equation (5.1) becomes

$$1 - \frac{1}{e_1} = \frac{q-1}{q},$$

so $e_1 = q$.

Let $D = (f^{-1}B_1)_{\mathrm{red}}$. Since

$$f^*B_1 = e_1 D = qD, \qquad B_1 \sim H, \qquad f^*H \sim qH,$$

we obtain $D \sim H$. Because H is primitive and every prime divisor has class a positive multiple of H, the reduced effective divisor D has exactly one prime component. Its inertia subgroup has order $e_1 = q$. By Lemma 2.4, this single inertia subgroup generates G. Hence $|G| = q$. On the other hand, Lemma 2.2 gives $|G| = \deg f = q^n$. Since $q > 1$ and $n \geq 2$, this is impossible. ☐

Proof of Theorem B. Let $n = 4$, and suppose that X is not isomorphic to $\mathbb{P}^4$. By Proposition 5.1, $i_X < n - 2 = 2$, so $i_X \leq 1$. Since X is Fano, its index is positive; hence $i_X = 1$. This contradicts Lemma 5.2. Therefore $X \simeq \mathbb{P}^4$. ☐

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